# Section 3.9

## Logarithmic Differentiation

(1) Review of Logarithm Properties and Derivatives

(2) Logarithmic Differentiation

# Review, Logarithmic Functions

## Definition of Logarithms

$$y = \log_b(x) \quad \Leftrightarrow \quad b^y = x$$

## Basic Properties

(I) $b^{\log_b(x)} = x$ and $\log_b(b^x) = x$.

(II) $\log_b(xy) = \log_b(x) + \log_b(y)$.

(III) $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$.

(IV) $\log_b(x^y) = y\log_b(x)$.

## Derivatives

$$\frac{d}{dx}(\log_b(x)) = \frac{1}{x\ln(b)} \qquad \frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

# Examples, Logarithms

(a) $\log_4(16)$

(b) $\log_4\left(\frac{1}{16}\right)$

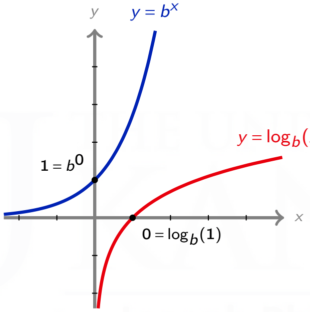
(c) $\log_{1/4}\left(\frac{1}{16}\right)$

(d) $\log_2\left(\sqrt[3]{2}\right)$

(e) $\log_2\left(\frac{1}{4\sqrt{2}}\right)$

(f) $\ln\left(\frac{x^2(x-1)}{\sqrt{x+1}}\right)$

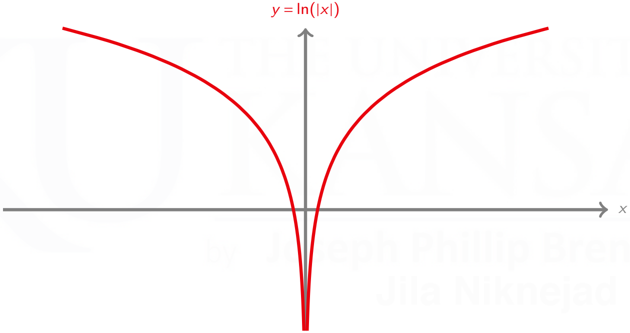
The domain of $\log_b(x)$ is the range of $y = b^x$, namely $(0,\infty)$.




So the identity $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ is valid only if $x > 0$.

**Question:** How can we extend the logarithm function to something whose derivative is $\frac{1}{x}$ for all nonzero $x$?

**Answer:** Define $f(x) = \ln|x| = \begin{cases} \ln(x) & \text{if } x > 0, \\ \ln(-x) & \text{if } x < 0. \end{cases}$




Then $\dfrac{d}{dx}(\ln|x|) = \begin{cases} \frac{1}{x} & \text{if } x > 0, \\ \frac{-1}{-x} & \text{if } x < 0. \end{cases}$

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x} \text{ for all } x \neq 0.}$$

# Logarithmic Differentiation

**Example 1:** Let $\boxed{y = f(x) = x^x.}$ Find the derivative $f'(x)$.

⚠ The rules $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(b^x) = b^x \ln(b)$ do not apply!:

- $f(x) = x^x$ is not a power function (the exponent is not a number)
- $f(x) = x^x$ is not an exponential function (the base is not a number)

Use the technique of logarithmic differentiation.

(I) Take the natural logarithm of both sides of the equation $y = x^x$ and use the Laws of Logarithms to simplify the expression.

(II) Differentiate the expression implicitly with respect to $x$.

(III) Solve for $dy/dx$.

# Logarithmic Differentiation

## Logarithmic Differentiation

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(II) Differentiate the expression implicitly with respect to $x$.

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# Logarithmic Differentiation

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In general there are four cases for exponents and bases:

(1) $\dfrac{d}{dx}\left(a^b\right) = 0$

(2) $\dfrac{d}{dx}\left([f(x)]^b\right) = b[f(x)]^{b-1} f'(x)$

(3) $\dfrac{d}{dx}\left(a^{g(x)}\right) = a^{g(x)} \ln(a) g'(x)$

(4) $\dfrac{d}{dx}\left([f(x)]^{g(x)}\right) = \left(g'(x)\ln(f(x)) + \dfrac{g(x)f'(x)}{f(x)}\right)[f(x)]^{g(x)}$

# Example 2, Logarithmic Differentiation

(I) Find the derivative of $y = (2x+1)^5(x^3-1)^3$.

(II) Find the derivative of $y = \left(\frac{x^2(x-1)}{\sqrt{x+1}}\right)^\pi$.

# Logarithmic Differentiation: Example 2:

(III) $y = x^{\sin(x)}$ :

(IV) $y = \sin(x)^{\arctan(x)}$ :

# Example 3

(I) Let $f(x) = \log_a(3x^2 - 2)$. For what value of $a$ is $f'(1) = 3$?

(II) On what interval(s) is the function $f(x) = \dfrac{\ln(x)}{x}$ increasing, decreasing, concave upward and concave downward?

# Example 4, Logarithmic Implicit Differentiation!

Consider the curve defined by the equation $y\sqrt{x^2+3} = x^y$.
Find the equation of the tangent line at the point $(1, \frac{1}{2})$.